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Improved energy level statistics for a family of billiards with analytic boundaries

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Abstract. The energy level statistics of a family of classically chaotic billiards with analytic boundaries is studied numerically with a high χ^2 confidence level for the final results, which go substantially beyond the quality of existing similar results. The hypothesis of Bohigas *et al* is strongly supported by the present results, showing that the level statistics of classically ergodic systems in the semiclassical limit can indeed be described by the Gaussian orthogonal ensemble of the random matrix theories, contrary to some doubts which have been recently raised e.g. by Wilkinson *et al*. Nevertheless, some small deviation of the numerical results compared with the Gaussian orthogonal ensemble is observed and shown, but it is believed to be still a deficiency of the statistical significance rather than of the model.

1. Introduction

The purpose of the present paper is to revise and refine the numerical calculations of the energy levels of a family of classically chaotic (numerically ergodic) billiards with analytic boundaries. A subset of such a class of billiards has previously been studied in Robnik (1984, hereafter known as paper 1). The main goal is to improve the statistical significance of the energy level statistics and to confirm the compatibility of the numerical results with the statistics of the Gaussian orthogonal ensemble (GOE) of random matrix theory (RMT).

During recent years there has been a number of works dealing with the numerical analysis of the energy levels of classically chaotic systems, with the goal of confirming our expectation that the spectral statistics should be correctly described by the statistics of the RMT. (See the review paper by Eckhardt (1988) and the numerous references therein.) Bohigas *et al* (1984) were the first to propose the conjecture that energy level statistics of classically ergodic (or more chaotic) systems with time reversal symmetry should be subject to the GOE statistics of the RMT. If there is no time reversal symmetry as in the case of chaotic systems in magnetic fields or in rotating systems, e.g. billiards, then the Gaussian unitary ensemble (GUE) statistics is the appropriate one.

These latter systems have been studied by Berry and Robnik (1986), Robnik and Berry (1986) for the case of the family of billiards with analytic boundaries as analysed by the present paper and having a magnetic point flux somewhere in the interior thereby breaking the time reversal symmetry. (We named them Aharonov–Bohm billiards.) Seligman and Verbaarschot (1986) studied the sextic oscillator in

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magnetic fields, whilst Wintgen and Friedrich (1986) and Delande and Gay (1986) dealt with the statistics of the hydrogen atom in a strong magnetic field. In the hydrogen atom in a strong magnetic field, however, the time reversal symmetry is restored in spite of the presence of the magnetic field just because the system has reflection symmetry and therefore obeys the GOE statistics rather than GUE. So, the diamagnetic Kepler problem belongs to the class of systems which are described by the GOE, like the billiard systems of the present paper. The role of the anti-unitary symmetries in the predictions of level statistics has been discussed in detail and for the general case in Robnik and Berry (1986) and in Robnik (1986). In the present paper we treat only billiard systems with time reversal symmetry, i.e. there are no magnetic fields, but the analysis of systems without anti-unitary symmetries is offered in a separate paper (Robnik 1992).

The motivation for the present work is the relatively poor statistical significance of the existing numerical studies, most of which are consistent with the statistics of RMT, but do not provide a numerical proof of the conjecture by Bohigas *et al* (1984). In addition, some doubts have been recently raised by Wilkinson *et al* (1991) as to the applicability of RMT statistics to chaotic Hamiltonians (with two degrees of freedom), based on the model studies of the banded random matrices.

The present paper provides additional strong support to the conjecture by Bohigas *et al*, since the statistical significance of the present numerical results goes substantially beyond the achievements of the existing works, namely by an order of magnitude.

2. The definition of the family of billiard systems with analytic boundaries

The plane billiards that we have chosen are described by the boundary curve in the complex w-plane which is a conformal map of the unit circle in the complex z-plane. Thus we choose the simplest but rich enough generic family given by

$$w = z + Bz^2 + Ce^{i\phi}z^3 \tag{1}$$

where $z = x + iy$, $w = u + iv$, while B and C are real parameters and ϕ is a real angle chosen in such a manner as to warrant the absence of all geometric symmetries such as reflection symmetry. A set of the parameter values B, C, ϕ thus determines a given bounding curve in the w-plane, which is the image of the unit circle $|z| = 1$ in the z-plane.

We have chosen twelve different sets of parameters defining twelve different billiard boundaries, as shown in table 1. Each of them has been carefully checked to be numerically ergodic by investigating the Poincaré map on the surface of section. As an example, in figure 1 we show the shape of the Africa billiard (no 1 in table 1). All billiards are non-convex shapes. In fact, the non-convexity of the boundaries is sufficient to warrant the chaotic nature of the billiards, since it breaks all Lazutkin-type caustics in the configuration space and the associated invariant curves in the phase space. This is intuitively obvious, but there is also a theorem by Mather (1982) supporting this view. Some small islands of stability in the vicinity of stable periodic orbits could, in principle, still be present, but our numerical evidence does indicate that the systems are all ergodic for all practical purposes. Therefore the islands of stability must be very small indeed, if they exist at all.

If there are no singularities in the boundaries then the boundary is analytic and the conformal mapping of the unit disc in the z-plane onto the billiard in the w-plane is one-to-one, i.e. it is bijective. This makes it possible to solve the eigenvalue

Table 1. The defining parameters B, C, ϕ for twelve billiards, including the area $\mathcal{A}$ and perimeter $\mathcal{L}$. The billiard no 1 is the so-called Africa billiard, introduced and extensively studied in Berry and Robnik (1986).

No	B	C	ϕ	$\mathcal{A}$	$\mathcal{L}$
1	0.2	0.2	$\pi/3$	3.7699	7.1012
2	0.1	0.2	$\pi/3$	3.5814	6.9218
3	0.1	0.3	$\pi/3$	4.0526	7.7082
4	0.1	0.2	$2\pi/3$	3.5814	6.9444
5	0.2	0.2	$2\pi/3$	3.7699	7.1990
6	0.15	0.2	$2\pi/3$	3.6599	7.0486
7	0.1	0.15	$\pi/3$	3.4164	6.6644
8	0.1	0.25	$\pi/3$	3.7934	7.2644
9	0.07	0.2	$\pi/3$	3.5493	6.8916
10	0.2	0.15	$\pi/3$	3.6049	6.8428
11	0.2	0.25	$\pi/3$	3.9819	7.4507
12	0.15	0.1	$\pi/3$	3.3772	6.5997

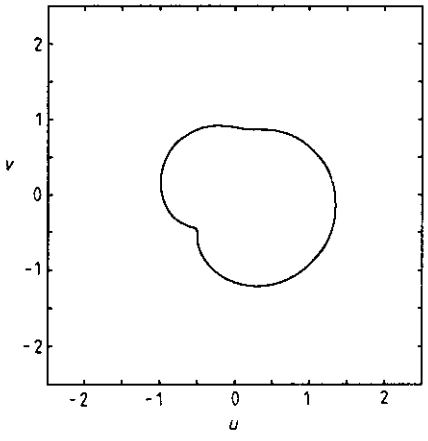

Figure 1. Example: the shape of the Africa billiard (number 1 from table 1). Each billiard boundary is the conformal image of the unit disc $|z|=1$ of the z-plane according to the equation (1), with the parameters given in table 1.

problem in the z-plane, where we have the basis of the Bessel functions, instead of solving it in the w-plane, where no basis is obvious. This is the main idea of the conformal transformation technique which we now describe.

3. The conformal transformation technique

This method has been devised in paper 1, and is also explained in Berry and Robnik (1986). Our eigenvalue problem is to solve

$$-\Delta_{uv}\psi = E\psi. \tag{2}$$

However rather than solving it in the uv coordinates we want to solve it in the transformed plane, the z-plane, in which the boundary becomes a unit circle, and the Laplacian is changed conformally, namely

$$\Delta_{uv} = \Delta_{xy}/J = (\partial^2/\partial x^2 + \partial^2/\partial y^2)/J \tag{3}$$

where the conformal factor J is the Jacobian of the transformation $w = w(z)$ as defined in (1), namely

$$J = |\, \mathrm{d}w/\mathrm{d}z \,|^2 \,. \tag{4}$$

Introducing the polar coordinates r and Θ in the xy-plane, through $z = r\exp(\mathrm{i}\Theta)$, the Jacobian $J(r,\Theta)$ can be written as

$$J(r,\Theta) = 1+4Br^2+9Cr^4+4Br\cos\Theta+6Cr^2\cos(\phi+2\Theta)+12BCr^3\cos(\phi+\Theta). \tag{5}$$

The eigenvalue problem (2) can now be rewritten in the z-plane, with the definition $\Delta = \Delta_{xy}$ (i.e. henceforth we omit the indices xy of the Laplacian in the z-plane), as follows

$$\Delta\psi + EJ(r,\Theta)\psi = 0 \tag{6}$$

where Δ can now also be written in the polar coordinates of the z-plane

$$\Delta = \partial^2/\partial r^2 + r^{-2}\partial^2/\partial\Theta^2. \tag{7}$$

Suppose the set $\{\varphi_j\}$ of eigenfunctions of $-\Delta$ on the unit disc forms an orthonormal basis. Let us expand our solution ψ

$$\psi = \sum_j c_j\varphi_j. \tag{8}$$

Then it follows that

$$\sum_i c_i(z_i^2\delta_{ij} - EJ_{ij}) = 0 \tag{9}$$

where z_j^2 is the eigenvalue of $-\Delta$ corresponding to the eigenfunction φ_j, and

$$J_{ij} = \langle\varphi_i \mid J \mid \varphi_j\rangle \tag{10}$$

are the matrix elements of the matrix **J**. By the definition of the matrix **U**,

$$(\mathbf{U}^{-1})_{ij} = z_i\delta_{ij} \tag{11}$$

and of the vector c,

$$(c)_i = c_i \tag{12}$$

equation (9) appears in the compact form

$$(\mathbf{U}^{-1}\mathbf{1}\mathbf{U}^{-1} - E\mathbf{J})c = 0. \tag{13}$$

Now, by multiplying by **U** from the left and defining $\mathbf{U}^{-1}c = k$, we obtain

$$(E^{-1}\mathbf{1} - \mathbf{U}\mathbf{J}\mathbf{U})k = 0. \tag{14}$$

The eigenvalues E are obtained from the solutions of the secular equation

$$\det \mid \mu\mathbf{1} - \mathbf{U}\mathbf{J}\mathbf{U} \mid = 0 \tag{15}$$

where $E = 1/\mu$, and $c = \mathbf{U}k$, with k being the corresponding eigenvector of the matrix **UJU**.

Our numerical method is the diagonalization of the matrix **UJU** in a truncated basis $\{\varphi_i\}, 1 \leqslant i \leqslant M$. Let us determine the basis $\{\varphi_i\}$. The eigenfunctions of (7) are the products of Bessel and trigonometric functions,

$$\varphi_{k,n} = R_{k,n} J_k(\gamma_{k,n} r) \begin{Bmatrix} \cos k\Theta \\ \sin k\Theta \end{Bmatrix} \tag{16}$$

with the normalization constants

$$R_{0,n} = [\sqrt{\pi} J_O'(\gamma_{0,n})]^{-1} \qquad R_{k,n} = [\sqrt{2}/\sqrt{\pi} J_k'(\gamma_{k,n})] \qquad k > 0. \tag{17}$$

Here J_k is the Bessel function of order k, $\gamma_{k,n}$ is the nth zero, and J_k' is the derivative of J_k. The eigenvalue of the Laplace operator (7) corresponding to the basis functions (16) is given by

$$-\Delta\varphi_{k,n} = \gamma_{k,n}^2 \varphi_{k,n}. \tag{18}$$

Now the eigenfunctions $\varphi_{k,n}$ are rearranged in order of increasing eigenvalues $\gamma_{k,n}^2$ and for this purpose we define the map $(k,n) \mapsto i = i(k,n)$ (and the inverse map $k = k(i), n = n(i)$.) Thus $z_i = \gamma_{k(i),n(i)}$ and $(\mathbf{U})_{ij} = \delta_{ij}/z_i$ (see equation (11)).

In order to work out the diagonalization procedure (14) we need to calculate the matrix elements J_{ij} of the matrix **J** (equation (10)), using the equation (5). Notice the very important circumstance that the parameter dependence of the matrix **J** on the parameters B, C, ϕ is known, since in calculating the elements (10) the integrals for the moments (various powers of r) with respect to the Bessel functions have to be (numerically) integrated only once, so that it is easy to study the parameter dependence of the energy levels for small matrices (M small), where the most of the CPU time is spent on the integration of the said moments, whilst for large matrices like in our case where $M = 2000$, this is less important because most of the CPU time is spent on diagonalizing the matrix (15).

Using this method I have calculated $M = 2000$ energy eigenvalues E for each of the twelve billiards defined in table 1. It should be noted that the numerical effort in calculating the twelve spectra is enormous: I have used a VAX8800 computer, and yet each diagonalization consumed about 30 h CPU time. Hence in total, the calculation of the twelve spectra 'cost' about *fifteen days of CPU time*, i.e. 360 h of CPU time!

Special attention has been paid to the various checks of the accuracy of the numerical diagonalization procedure. The size M of the matrices has been gradually increased, starting with $M = 500$, in order to confirm, that the energy levels converge to their exact values. In this manner it is found that for $M = 2000$ the first 550 levels are more accurate than 1% of the mean energy level spacing, and thus suitable for the significant statistical analysis.

Next the Weyl-type formula

$$\mathcal{N}(E) = \frac{\mathcal{A}E}{4\pi} - \frac{\mathcal{L}\sqrt{E}}{4\pi} + \frac{1}{6} \tag{19}$$

for the (cumulative) number $\mathcal{N} = \mathcal{N}(E)$ of levels below the given energy E has been applied for each billiard to see that the lowest 500 levels do have the correct average behaviour. One example, namely for the Africa billiard (no 1 in table 1), is shown in figure 2.

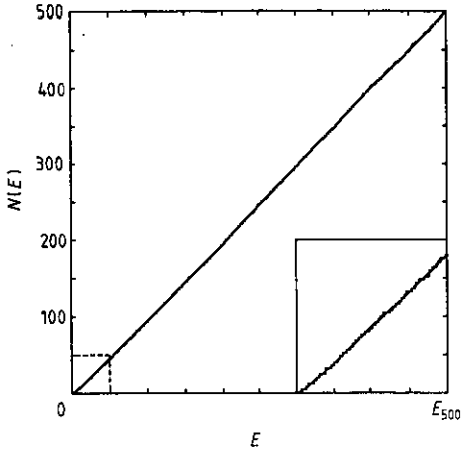

Figure 2. The spectral staircase for the lowest 500 levels of the Africa billiard (no 1 in table 1), and the theoretical curve according to the Weyl-type formula from the equation (19), with area $\mathcal{A}$ and perimeter $\mathcal{L}$ also given in table 1.

4. The results

The object of our numerical analysis is the spectral staircase, i.e. the function $N(E)$, which by definition is the number of energy levels not larger than the energy E.

In calculating the energy level statistics we have carefully performed the unfolding procedure (see e.g. the review paper by Bohigas and Giannoni (1984)), which means that the average behaviour as given by the Weyl-type formula (19) has been eliminated in the energy spectrum of each billiard given in table 1. Explicitly, this means the following transformation: The function $x(E) = \mathcal{N}(E)$ is used to map the spectrum $\{E_j\}$ onto the set of numbers $\{x_j = x(E_j)\}$,

$$x = x(E) = \mathcal{N}(E) \qquad x_j = x(E_j) = \mathcal{N}(E_j) \tag{20}$$

such that the spectral staircase $N(x)$ is now given as the sum of the linear average part plus the oscillating contribution $\tilde{N}(x)$, i.e.

$$N(x) = x + \tilde{N}(x). \tag{21}$$

Thus when we talk about the spectral statistics, we mean the statistics of the unfolded spectral staircase $N(x)$ as defined in equation (21).

Before the statistical analysis has been performed we have discarded in each spectrum the lowest fifty levels, and considered the five hundred levels from 51st to the 550th, in order to keep the 1% accuracy of the mean level spacings and at the same time have the levels as close as possible to the semiclassical regime. (In other words, the lowest 50 levels are considered as far from the semiclassical behaviour, and would therefore spoil the significance of the statistical analysis, and this is the reason for dismissing them.) Such unfolded and selected spectra are the subject of the statistical analysis of the present paper. For the sake of simplicity we will henceforth call x energy.

We consider three types of spectral statistics. The first is the level spacings distribution $P(S)$, defined as the probability density, such that the probability that the level spacing S lies in the interval $(S, S + \mathrm{d}S)$ is given by $P(S)\mathrm{d}S$. The GOE theory is closely approximated by the Wigner distribution, namely

$$P_{\mathrm{GOE}}(S) = (\pi/2)S\exp(-\pi S^2/4). \tag{22}$$

Next the rigidity $\Delta(L)$ is the local average least squares deviation of the spectral staircase $N(x)$ from the best-fitting straight line, over an energy range x of L mean level spacings. Thus

$$\Delta(L) = \left\langle \min_{(\alpha,\beta)} \frac{1}{L} \int_{-L/2}^{+L/2} \mathrm{d}\epsilon [N(x+\epsilon) - \alpha - \beta\epsilon]^2 \right\rangle. \tag{23}$$

The averaging indicated by $\langle \ldots \rangle$ is over each individual spectral stretch, and also over the ensemble of spectral stretches corresponding to various billiards of the given set. The GOE theoretical formula is asymptotically given by (see (Berry 1985))

$$\Delta_{\mathrm{GOE}}(L) \to (1/\pi^2) \ln L - 0.00695 \qquad L \gg 1. \tag{24}$$

The third spectral statistics is the cumulative level spacings distribution $W(S)$, i.e.

$$W(S) = \int_0^S \mathrm{d}t P(t) \tag{25}$$

where $P(S)$ is the level spacings distribution defined earlier.

Let us look at the results of the statistical analysis of our spectra. Two sets of spectra are considered, both of them being a composite of several spectra. The first one consists of the first three billiards from the table 1 ('trio'), and the second one consists of all the twelve billiards given in table 1 ('complete set').

The trio is thus a composite of $3 \times 500 = 1500$ levels. The results are excellent as seen in figures 3(*a*)–(*c*). In each of these figures we show three theoretical curves, namely the Poissonian (characteristic of integrable systems), the GUE curve and the theoretically expected GOE curve. It is readily seen that the agreement with theory is excellent and certainly better than in the existing data. The cumulative spacings $W(x)$ closely follow the theoretical curve of GOE (figure 3(*a*)), and so does the histogram in figure 3(*b*). The Δ-statistics shown in figure 3(*c*) is consistent with the theoretical curve of GOE. The saturation shown in figure 3(*c*) can be fully understood in terms of the semiclassical theory of spectral rigidity by Berry (1985): The saturation should start at $L_{\max} \approx 40$, and the saturation value should be $\Delta_\infty \approx 0.2$. Finally, the χ^2 test has been performed for the cumulative spacings distribution $W(S)$, with the result that the confidence level is 93%. We have also calculated the second moment of the histogram $P(S)$, with the result $\sigma^2 = \langle S^2 \rangle - 1 = 0.292$ (for the 80-bin histogram), whilst the theoretical value for the GOE is 0.286 ± 0.015, so that the numerical result is well consistent with the GOE value.

Now we consider the results for the complete set of billiards from table 1, and present the results in the same manner as for the trio, as shown in figures 4(*a*)–(*c*). This sample consists now of 6000 levels. Figure 4(*a*) shows that the agreement of the cumulative spacings distribution with the theoretical curve is now not so good as for the trio, which is a demonstration of the extremely slow (and oscillatory) convergence of statistics. The χ^2 confidence level for $W(s)$ is now somewhat smaller, namely 80%, although the histogram with 80 bins in figure 4(*b*) is considerably better than in the trio, as judged by naked eye. Finally, the Δ-statistics is well behaved and is almost the same as for the trio. The σ^2 is now 0.298, which is consistent with the GOE theoretical value.

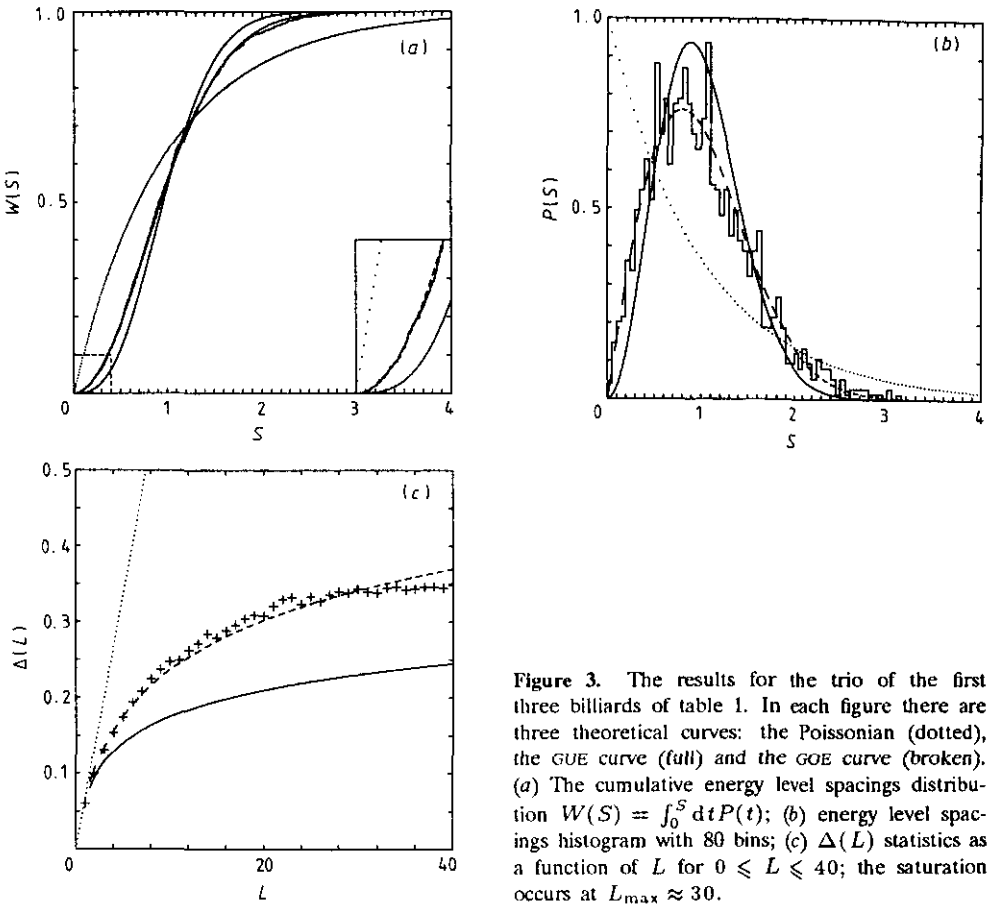

Figure 3. The results for the trio of the first three billiards of table 1. In each figure there are three theoretical curves: the Poissonian (dotted), the GUE curve (*full*) and the GOE curve (broken). (*a*) The cumulative energy level spacings distribution $W(S) = \int_0^S \mathrm{d}t P(t)$; (*b*) energy level spacings histogram with 80 bins; (*c*) $\Delta(L)$ statistics as a function of L for $0 \leqslant L \leqslant 40$; the saturation occurs at $L_{\max} \approx 30$.

5. Conclusions and discussion

In my view the results of the present work do provide strong additional evidence in support of the conjecture by Bohigas *et al* (1984) that the energy levels of classically ergodic (or more chaotic) billiards obey the GOE statistics of the RMT, if they possess time reversal symmetry, or some other anti-unitary symmetry. (Recently some doubts have been raised by Wilkinson *et al* (1991) as to the validity of the said conjecture.) This assertion applies both to the trio ensemble of the first three billiards of table 1, as well as to the complete set of the twelve billiards of table 1. The assertion is consistent with the semiclassical considerations, e.g. with Berry's (1985) theory of spectral rigidity: As $\hbar$ goes to zero, the fluctuations vanish, the border of the saturation regime $L_{\max}$ tends to infinity and so does the saturation level Δ_∞, so that gradually the entire range $0 \leqslant L \leqslant \infty$ displays the universal behaviour of the Δ-statistics as given in equation (24).

At the same time the present work shows that the convergence with increasing size of the ensemble of energy levels is extremely slow, reflected in the fact that the results for the twelve billiards are not of better quality than the results for the trio. Most notable is the deviation of the cumulative spacings distribution $W(S)$ from the GOE curve for the complete set of twelve billiards (figure 4(*a*)): the numerical

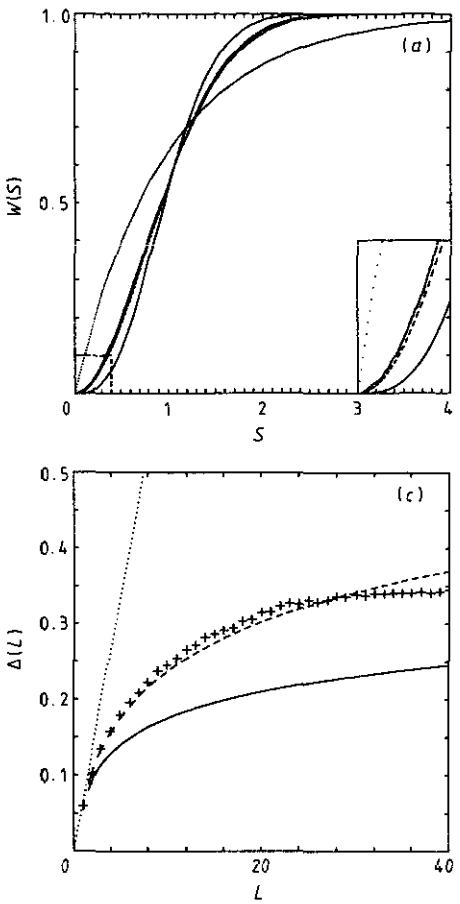

Figure 4. The results presented as in figure 3 but now for the complete set of twelve billiards as given in table 1. In each figure there are three theoretical curves: the Poissonian (dotted), the GUE curve (full) and the GOE curve (broken). (*a*) The cumulative energy level spacings distribution $W(S) = \int_0^S dt P(t)$; (*b*) energy level spacings histogram with 80 bins; (*c*) $\Delta(L)$ statistics as a function of L for $0 \leqslant L \leqslant 40$; the saturation occurs at $L_{max} \approx 30$.

staircase deviates slightly from the theoretical curve for small S, whereas in the case of the trio there is excellent agreement.

This deficiency is thus believed to be a manifestation of the slowness of the convergence of the statistical measures rather than a proof of a definite deviation of the measures from the GOE theoretical curves. This slow convergence is also a clear demonstration that the results presented in this paper are just at the border of the reasonable size of ensembles suitable and justified for the numerical analysis, given the capacity of present day computers.

There is no doubt that the slow convergence has something to do with the classical periodic orbits and their oscillatory contributions to the spectra in the spirit of the Berry's theory of spectral rigidity. It would be quite odd indeed, if the eigenfunctions of classically ergodic (or more chaotic) systems would display scar phenomena (Heller 1984, 1991) which are the deviations from Gaussian randomness, but no associated phenomena in their energy spectra would exist. This is implicit in the early and fundamental theory of Gutzwiller (see the review by Eckhardt (1988) and the references therein). The explanation is that the GOE behaviour is reached when $\hbar \to 0$ or, equivalently, when the energy goes to infinity, i.e. the infinite number of levels are considered in each individual spectrum of each individual billiard.

There could be, in principle, two further theoretical reasons for a deviation of

statistics from the GOE predictions. One reason could be the existence of an approximate reflection symmetry (the billiard shape is close to one with a reflection symmetry) or some other discrete geometrical symmetry. This has been largely eliminated by the careful choice of the parameters B, C, ϕ, such that the spectral statistics of each individual billiard was clearly GOE.

We can, however, report peculiar energy level statistics for two billiards which have been studied but are not shown and considered in this work. These are $B = 0.2, C = 0.1, \phi = \pi/3$ and $B = 0.1, C = 0.1, \phi = \pi/3$. Both of them are certainly numerically ergodic (no islands of stability have been detected), and also, at first glance they are sufficiently far from having a discrete geometric symmetry. Yet, they do display energy level statistics characteristic of two randomly superposed GOE sequences, rather than one. For this reason they have been dismissed. This fact might imply that the transition from one GOE to two superposed GOE statistics might be smooth for finite spectral stretches, and calls for caution. It could be one reason for the fact that the results for the trio are better than the results for the complete set, just because some of the billiards of the complete set of table 1 might not be sufficiently far from having some geometric symmetry, the reflection symmetry or the parity, even though the overall behaviour for each of them was clearly GOE, as mentioned above. It is proposed to carry out a quantitative analysis of the transition of the statistics from GOE to 2 × GOE as the geometrical shape gradually approaches the shape with reflection symmetry (or inversion symmetry).

Another reason in principle for deviations from GOE would be the existence of small islands of stability, i.e. broken ergodicity of the billiard system. This has also been eliminated by the careful study of the Poincaré maps (bounce maps) for each of the twelve billiards. So, it is our belief that the deviation from GOE is predominantly a genuine semiclassical effect of classical periodic orbits, which make the convergence of the statistics very slow.

In conclusion, the present work supplies firm evidence in support of our view that in the semiclassical limit $\hbar \to 0$ the energy level statistics of classically ergodic (or more chaotic) Hamiltonian systems is correctly described by the GOE statistics, if there is time reversal symmetry or some other anti-unitary symmetry.

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